Labour in the Circular Economy: the catalyst towards sustainable development

Julie Metta[1]

[1] *Corresponding author. Research Group Sustainable Development, Faculty of Economics, KU Leuven, Parkstraat 47 bus 5300, 3000, Leuven, Belgium. Email: julie.metta@kuleuven.be*

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To the reviewer from FAERE Conference 2020: *Thank you for taking the time to review this manuscript. Please kindly note that this version is a pre preliminary version not yet completed, and it is the first time that this work is presented to a conference. Many thanks for your kind time and understanding.*

Abstract

This study analyses the circular economy as an alternative to cope with both the exhaustibility of resource, and the pollution induced by improper waste management. I shape a two-sector growth model in a simple closed economy. The waste stock is assumed to be inappropriately managed, and to negatively affect labour productivity. (e.g. January 2019, in Rome, Italy, a landfill burned forcing workers to stay home for several days - The Independent, 2019). I investigate the optimal allocation of labour between the production and the circular sector to sustain development while limiting environmental impact caused by waste on labour. I model changes in economies due to an endogenous choice of labour in the supply and I analyse under which conditions the circular economy can generate endogenous growth. My approach is innovative as I emphasise the role of the labour in the circular economy, illustrating the important role of the circular sector to sustain economic development. The results lead to a full circular economy model for the management of resources. The model offers a new approach for substitutable development of supply chains.

Key words

Labour productivity; damage function; endogenous growth; pollution; recycling

1 Introduction

The market failures related to the current linear economy, also known as Take-Make-Dispose, have led to pollution, depletion of exhaustible resources and production of waste. An increasing number of countries is dealing with negative externalities related to inadequate waste management. Improper waste management leads to several health and environmental hazard such as air and soil pollution, land occupation, and even hazardous blazes. Beyond the health damages on workers and their family, poor waste management constrains people to move further away from the waste management facilities (Giusti, 2009; Ferronato & Torretta, 2019), creating further distance between labour force and job sites. Eventually, improper waste management decreases labour productivity. Even in developed cities such as Rome, labour productivity has been negatively impacted by the externalities induced by waste management. To avoid this loss in labour productivity, circular economy could be promoted as a way to transition from a highly polluting economy to one that is more sustainable. Circular economy aims to increase resource reuse and economic performance while reducing negative externalities on the environment and on society (Wijkman & Skånberg, 2015). The circular economy model relies on concepts such as: reduce, reuse, recycle, redesign, remanufacture, recover, share. Although the concept of circular economy is often mentioned by practitioners[1] and scholarly acknowledged in terms of resource management, few papers study the impact of inadequate resource and waste management in the linear economy on labour productivity.

This research evaluates the impact on labour productivity of, on the one hand, the negative externalities induced by waste management, and on the other hand, the circular sector, apprehended as an alternative to the linear production sector. A two-sector Schumpeterian model of growth is shaped, with the first sector producing goods and waste from resources, and the second sector (the circular economy sector) producing recycling resources from the stock of waste, to be used by the first sector. With the increasing policy interest towards circular economy, such study, analysing the potential of the circular economy sector to enhance labour productivity while reducing pollution, is essential for policy makers to develop adequate instruments.

Ellen MacArthur Foundation (2018) reported that the circular economy could make "cities more liveable, reducing emissions of fine particulate matter by 50%, emissions of greenhouse gases by 23%, and traffic congestion by 47%, by 2040 [...] (and help) save businesses and households " money. While practitioners mention the benefit of circular economy on labour productivity mainly due to the reduction of pollution induced by inadequate waste management, no study has yet evaluated the impact of circular economy on labour productivity.

The model presented in this research proposal tends to substantiate the literature on circular economy and its impact on labour productivity, by shaping a two sector growth model in a simple closed economy. I investigate the optimal allocation and productivity of labour between the first sector, which provides the main production, and the circular sector, to sustain development while limiting environmental impact caused by waste on labour. The social planner's solution suggests the optimal distribution of the labour supply between the main production function and the circular sector. Through this approach, I determine the optimal 1) circular rate to sustain development while limiting environmental impacts by waste on labour, 2) skill premium to allow the achievement of the circular economy.

The first sector is represented as a standard extracting production function, which exploits exhaustible resources and creates the economic outputs but also waste which accumulates in a stock, negatively impacting the labour productivity. In this model, exhaustible resource scarcity, pollution damage induced by the stock of waste and labour productivity

[1] Practitioners are government, business and civil society

are endogenous components in the overall production. The exhaustion of the resources impacts the first sector of the economy, leading to degrowth as the material input becomes scarce. The stock of waste creates various negative externalities. The first one is the use of space. No household wishes to live close to a waste treatment facility and thus, when landfills extend or when new waste treatment facilities are built, labour force moves out, increasing the average commuting time to the workplace. Secondly, waste management facilities are a source of air pollution, from the gas emitted and from the accidental blazes. Beyond health related issues impacting labour productivity[2], more severe accidents may occur due to improper waste management. In 2018, in Rome citizens were asked to stay inside for few days due to the toxic smokes emanating from the nearby ignited landfill[3]. Such events have negative consequences on labour productivity.

The second sector of this model represents the circular economy sector which uses the stock of waste created by the first sector as input. The second sector provides an alternative material input to the first sector, partially solving the issue linked with resource exhaustion. This circular economy sector impacts labour productivity in two ways. Firstly, such as the education sector does, the circular economy offers an alternative and gratifying sector for workers, whilst taking labour force out of the market for the first sector. Secondly, by exploiting the waste generated by the first sector, the circular economy sector reduces the negative externalities affecting labour productivity. The overall impact of the circular sector on the economy is to be evaluated in by this model.

In developing such a model, I investigate whether sustainable development is driven by an increase in endogenous circular economy (or put differently by a reduction in both the use of exhaustible material and its induced pollution on labour productivity). The model offers a new approach for the substituability of exhaustible and recycled materials. I investigate the optimal allocation and productivity of labour between main production and circular sector to sustain development while limiting environmental impact caused by waste on labour.

The outcomes of the study supports policies to promote circular economy for labour productivity enhancement. Policy stakeholders might consider financial instruments to internalise the damages linked with waste management and to intensify the circular economy sector.

2 The junction of three main strands of the literature (*Literature review*)

This research proposal builds on three strands of the literature. First, it relates to the literature on circular economy related to resource depletion. Authors investigate circular economy as a way to provide substitute for exhaustible resources since mid-1970s' (Hoel, 1978; Lusky, 1976; Hoel, 1978; De Beir et al., 2010; Di Vita, 2001; Pittel et al., 2010; Lafforgue & Rouge, 2019; Zhou et al., 2019; Fagnart & Germain, 2011; Sorensen, 2017; Boucekkine & El Ouardighi, 2016). Di Vita (2001); Pittel et al. (2010); Lafforgue & Rouge (2019) model the circulation of materials, considering waste either as a flow or as a stock. Lafforgue & Rouge (2019) do not consider a stock of waste which can accumulate and thus create pollution, they introduce the circular economy sector to uniquely overcome the flow of waste. Zhou et al. (2019) considered also the circular economy has a way to feed the production function with non-exhaustible resources and as they consider no loss in the process, there is no need for injection of exhaustible resources. Fodha & Magris (2015) study the impact of circular economy on economic welfare and resource exhaustion with overlapping generations. De Beir et al. (2010) concede that the potential of circular

[2] *Rome blanketed in smoke after huge fire breaks out at disposal plant amid waste crisis* (2018)
[3] *'Disgusting dumpsters': Rome garbage crisis sparks health fears* (2019)

economy is similarly not bounded by the stock of waste and produces unlimited material supply to the economy. De Beir et al. (2010) consider circular economy as dependent on the output produced in the last period to run the economy. The model presented in this proposal differs from most of the studies as the available input for the circular economy depends on the stock of waste generated by the first sector of the economy. Also, and to make the model more realistic, losses (due to entropy) are considered in all sectors. Waste is generated through the extraction of exhaustible materials and reduces by its partial re-injection in the economy as recycled material.

Second, this paper relates to the impact of pollution on the economy. In the model of Pittel et al. (2010), and to achieve a balanced growth, policy arguments are needed to overcome the waste flow through recycling, as waste is considered as a not harmful stock of material which can be used as input and thus become recycled material. Similarly, the conclusions of Fagnart & Germain (2011) are pessimistic as they fail to consider pollution damage from their available flow of waste. These results are mainly explained as no damage occurred by the waste accumulation. However, Hoel (1978) and Lusky (1976) provided more optimistic results when considering circular economy to control for pollution induced by exhaustible material extraction and use. Circular economy is considered as an alternative to cope with the pollution induced by natural resource extraction on the environment (Lusky, 1976; Hoel, 1978; Fagnart & Germain, 2011; Sorensen, 2017; Boucekkine & El Ouardighi, 2016). Fagnart & Germain (2011); Sorensen (2017); Boucekkine & El Ouardighi (2016) introduced a secondary sector, the circular sector, to overcome the harming flow of waste from exhaustible material. Bretschger & Pattakou (2019) modelled a damage function, representing the pollution, which impacts the capital stock needed for the main production. Following this strand, I model a damage function caused by the stock of waste which accumulates from the use of exhaustible material and reusable material used in the main production function. I assume a current flow of waste to be taken from a stock of waste. Waste need to be consider both as a flow, for the material balance constraint (i.e., the substitutability of exhaustible and recycled material) and as the stock to account for its negative externalities. In my model the circular economy is considered as an alternative to resource exhaustion, and to the pollution induced by improper waste management.

Most of the time, the pollution induced by waste, is considered to impact the environment through the reduction of capital (Hoel, 1978; Fagnart & Germain, 2011; Boucekkine & El Ouardighi, 2016; Sorensen, 2017; Bretschger & Pattakou, 2019). In most of the study on the circular economy, the role of labour is neglected. Only Lusky (1976) evaluated the negative impact of waste on labour but do not consider constrain the non-recycled material as exhaustible material. My approach focus on the impact of the inappropriate waste management on the labour productivity. This impact is caused by the stock of waste which accumulates from the use of material in the economy and by the entrance of a new sector: the circular economy sector. The model is based on labour productivity and labour is the crucial input of the two production sectors. Final goods are produced from an exogenous technology level, labour and two types of material inputs: exhaustible resources and recycled material. The growth of the waste stock implies negative impacts on labour productivity. I consider labour to be either devote to either the main production function or to the circular economy sector.

Last but not least, the paper engages with the literature on labour productivity as a determinant factor of growth. The circular economy is modeled as a second sector impacting labour productivity, following the model developed by Rada (2007). Parallels can be drawn between the role of the circular economy and the education sector to enhance labour productivity (see for example Olley & Pakes (1992), Griliches (1997) and Acemoglu & Autor (2012)). As mentioned in the early work of Griliches (1997) on the role of education to enhance productivity, both sectors diverge a part of labour from the main production sector but increase the productivity of the labour for the economy. While for the education sector, the productivity is transferred as better qualified labour force for the production sector, in the circular economy, the productivity is improved through pollution reduction. As highlighted in the work of Lebedinski & Vandenberghe (2014), several factors (in this peculiar paper it is Education) enhance labour productivity. The model is estimating the impact of the circular economy sector to enhance labour productivity. The circular economy sector also intervenes as determinant of growth, through the enhancement of labour productivity and the provision of secondary material for the production sector. Similarly to the work on the contribution to labour productivity conducted by Knowles & Owen (1997) and more recently by Madsen & Murtin (2017), the model demonstrates the important role of the circular economy sector on labour productivity for economic growth.

3 The impact of inadequate waste management on labour productivity (*Conceptual framework*)

None of the studies mentioned in the previous section accounts the circular economy sector for solving exhaustibility of resources, reducing the waste stock and the negative externalities related, and for impacting labour productivity. The main role of this paper is to investigate the optimal labour productivity to sustain economic development while limiting environmental impacts of waste. This study is novel as it analyses the circular economy as an alternative to cope with both the exhaustibility of resources, and the pollution induced on labour productivity by improper waste management of these resources. From the model developed by Lafforgue & Rouge (2019), a new model is set up including losses in the material use and a damage function impacting labour productivity. This new model allows original results supporting policies towards the circular economy.

To model the economy, I construct a two-sector (Z and Y) growth model in a simple closed economy which relies on assumptions about aggregate levels of technology, resource and labour. The first sector, Y, is, as per say, a production sector, producing goods directly consumed by consumers. The second sector, Z, is the circular sector, which tackles waste created by the first sector. The model developed presents the circular economy sector as all together reuse, recycle, repair and share, which fully ascertains the assumption that the circular sector reduces the waste stock. Labour and waste are the two main components of the circular sector. The waste stock is assumed to be inappropriately managed, and to negatively impact labour. Exhaustible resource, pollution damage, recycled material used and the labour productivity are endogenous components in the overall production. I model changes in economies due to an endogenous choice of labour supply and investigate under which conditions the circular sector can generate endogenous growth. The model relies on assumptions about aggregate levels of material use and loss, and externalities linked with waste and impacting labour. My approach is innovative as I emphasize the role of labour, illustrating the important role of the circular economy for sustainable economic development. The novel model offers a new approach for the substituability of exhaustible and recycled materials.

- $y(t)$, the production sector produces goods directly consumed by consumers such as $y(t) = f(A_y(t), m(t), l_y(t))$ In a later stage, f can be considered a Cobb-Douglas function such as $y(t) = A_y(t)\, m(t)^{\theta} l_y(t)^{1-\theta}$. This sector is driven by labour, material and an exogeneous factor of productivity. As illustrates by Figure 12, the production sector Y consumes material initially extracted from an exhaustible stock of resources. This production sector Y produces waste which accumulates into a stock which damages the labour. What if the waste accumulation has a NON negligible impact on capital? Check in a second time.

- $A_y(t)$ is the exogenous total factor productivity (TFP) of Y and includes both energy and technology supply.The potential growth of $A_y(t)$ over time could be evaluated in a second time.
- $l_y(t)$ is the labour supply to the production sector.
- $m(t)$ captures the material inputs used for the production of goods. Both exhaustible and recycled materials can be used as a substitute for material in the production function such as $m(t) = A_x(t)\,x(t) + A_z(t)\,z(t)$. $A_x(t)$ (respectively $A_z(t)$) represented the exogenous technology needed to transform the exhaustible (recycled) material into material available to input in the main production function. We consider $A_x(t) > A_z(t)$. This inequality of material productivity implies that the production sector Y prefers to use exhaustible material $x(t)$ over the recycled material $z(t)$. A different relation between $A_x(t) \neq A_z(t)$ could mean that exhaustible material is taxed or recycled material is subsidized. Taxing $x(t)$, respectively $s_x(t)$ would make it somehow scarce from a market perspective and would allow more circularity of material. Those possibilities will have to be further investigated in a second time.
- $x(t)$, exhaustible material is extracted from a finite stock of exhaustible resource with stock $s_x(t) \geq 0$ and $\dot{s}_x(t) = -x(t) \leq 0$
- $s_w(t)$ the stock of waste which accumulates from the production sector.
- $z(t)$, the recycled material, say differently, the output obtained from our second sector: the circular sector Z. This output $z(t)$ is used as material $m(t)$ to be input in the first production function Y. The circular sector is a function of labour, waste available and an exogenous factor of productivity. $z(t) = h(E(t), w(t), l_z(t))$. In a later stage, h can be considered a Cobb-Douglas function such as $z(t) = E(t)\,w(t)^{\nu} l_z(t)^{\gamma}$.
- $E(t)$ is the exogenous total factor productivity (TFP) of Z and is considered as unlimited renewable energy and technology. The model will investigate the potential growth of $E(t)$ over time in a second time.
- $l_t(t)$ is the labour input in the circular sector. As detailed in the introduction, labour is the crucial source of waste management and research and development in this sector is negligible. Thus, I assume the circular sector not to be technology intensive. Most of the technology needed to recycle waste is already existing and there is no need for additional capital to be invest in technology to recycle, comforting the assumption.

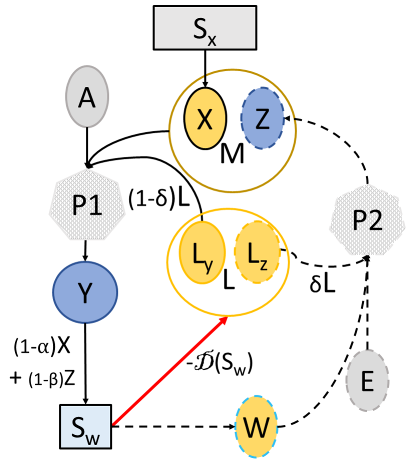

Figure 1: Representation of the overall model

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

- $w\,(t)$, is the flow of waste used to produce recycled materials. $w\,(t)$ is extracted from the stocks of waste $s_w\,(t)$. This stock of waste available for the circular sector is constituted either by the exhaustible material $x\,(t)$ or by the stock of waste from the recycled material $z\,(t)$. The waste stock dynamics follow $\dot{s}_w\,(t) = (1-\alpha)x\,(t) + (1-\beta)z\,(t) - w\,(t)$. α (respectively β) captures the loss of the system with regard of the exhaustible (recycled) resources. When not all resources used can be recovered then $\alpha > 0$ (respectively $\beta > 0$). It is rational that more loss occur on the exhaustible resources which are not circular by design and thus $\alpha \geq \beta \geq 0$). When all products created from recycled material $z(t)$ are recycled per design $\beta = 0$. If we assume no loss in the system then α can be considered null $\alpha = 0$. All exhaustible resources extracted accumulate in the waste stock and are responsible for the damage on the stock of labour. Similarly, because all materials from the circular sector are designed to be fully recyclable, there is no loss in the circular sector and β is assumed to be null $\beta = 0$. The stock of waste from the recycled material can be written $\dot{s}_w\,(t) = x\,(t) + z\,(t) - w\,(t)$.

- Inappropriate waste management leads to pollution leakage into the water, soil and atmosphere, which impacts the amenities for labour. The damage function is considered to only impact labour but it is present in the utility as per say. Indeed, the damage induced by the improper waste management is firstly spacial, workers move out from the area where the waste is manage and need longer to commute, decreasing their productivity. When incidents occur, workers are constrainted to stay at home, decreasing once more their productivity, but they keep on receiving their salary. A function of the stock of waste $\mathcal{D}\,(s_w(t))$ captures those negative externalities. $\mathcal{D} = L\frac{2}{\iota}\left(\frac{\iota}{2} - e^{-\frac{1}{s_w(t)}}\right)$ is continuously differentiable and convexe for $s_w(t) \geq 0$, with $\lim\limits_{s_w(t)\to+\infty} \mathcal{D}(s_w(t)) = \frac{L}{\iota}$ and $\mathcal{D}(0) = L$. Another option for the damage function would be: $\mathcal{D}\,(s_w\,(t)) = \frac{L}{\iota}(1-e^{-s_w})$ this way when $s_w\,(t) \geq 0$, $\mathcal{D}\,(0) = 0$, and $\mathcal{D}'\,(0) = L$ and $\mathcal{D}\,(\infty) > 0$ and $\mathcal{D}'\,(\infty) > 0$. This way $\mathcal{D}(s_w(t))$ would be concave for $s_w\,(t) \geq 0$.

- $l\,(t) = l_y\,(t) + l_z\,(t)$, labour is the labour available in the economy. $l_y\,(t)$ is used for the production sector to produce economic output and $l_z(t)$ used in the circular sector to produce recycled materials.

- $\delta(t)$ is the share of labour dedicated to circular economy, i.e., $l_y(t) = (1-\delta_t)l(t)$ and $l_z(t) = \delta_t l(t)$. When there is no circular sector $\delta_t = 0$, else I assume δ_t to be positive strictly inferior to 1 $0 \leq \delta_t < 1$.

  Labour is negatively affected by the waste stock from exhaustible material. The negative impact of waste on labour is especially true in developing countries where waste is often improperly managed. Additionally, landfills occupy space reducing amenities access for labour. I consider only the stock of waste from the exhaustible materials to negatively impact the labour. I assume no damage on labour from the circular sector. The stock of waste from recycling material is assumed not to negatively impacted the labour such that the impact comes exclusively from the stock of $s_{wx}(t)$. The labour function reads: $l(t) = \mathcal{D}(s_{wx}(t)) = L\frac{2}{\iota}\left(\frac{\iota}{2} - e^{-\frac{1}{s_{wx}(t)}}\right)$. If we consider : $\mathcal{D}(s_{wx}(t)) = \frac{L}{\iota}(1 - e^{-s_{wx}(t)})$ then we have to consider $l(t) = L - \mathcal{D}(s_{wx}(t)) = L - \frac{L}{\iota}(1 - e^{-s_{wx}(t)})$. L captures the maximal amount of labour available at any time. Indeed, some categories of people (mostly wealthy) would never be affected by the waste stock externalities. The social planner make a trade off between the wealth and the labour supply to dedicate to the circular sector. ι captures the maximum labour productivity loss induced by the pollution, and it is assumed that $1 < \iota$. The damage could also enter the utility function of consumer and force the labour to reduce their supply. This alternative would be more elegant, in a second time check if same results

$$
\begin{aligned}
l(t) &= \mathcal{D}(s_w(t)) \\
l_y(t) &= (1-\delta(t))\, l(t) \\
l_z(t) &= \delta(t)\, l(t) \\
0 \leq \delta(t) &< 1 \\
0 < l(t) &\leq L \\
0 < l_y(t) &\leq L \\
0 \leq l_z(t) &< L \\
l^0 &= l_y^0 = L
\end{aligned} \tag{1}
$$

- The economy consists of representative consumers maximizing their utility. The utility function of the consumer $u(c(t))$ is of the Constant Relative Risk-Aversion Utility (CRRA) form and considered as standard life time without constraints: $u(c(t)) = \frac{c(t)^{1-\sigma}-1}{1-\sigma}$ for $\sigma \neq 1$. The consumer in this economy owns physical capital (labour) supplied to the two sectors, and consumes the goods produced by the main production function. $c(t)$ is a function of the output from the main production function and of the wage provided by one of the two sectors $c(t) = g(y(t), \omega(t))$.
In a second step, the damage could also enter the utility function and force the labour to reduce their supply. This alternative would be more elegant.
In a later stage, economic capital owned by the consumer could be considered as input of the two production function. In that case, a return on capital r(t) would be added to the consumer function such that: $c(t) = g(y(t), \omega_i(t), r(t))$.

- $\omega_i(t)$ represented the wages, $i \in \{y; z\}$ The wages might differ between the first sector (y) and the circular sector (z). The model predicts the skill-premium $log(\omega_y)/log(\omega_z)$ between the two sectors.

- The social planner maximizes consumer's inter-temporal utility subject to the production functions of the two sectors ($y(t)$ and $z(t)$) and the stocks of exhaustible resources $s_x(t)$, wages $\omega_i(t)$ and labour available l . The use of exhaustible material $x(t)$ induces damage on the stock of labour l thus the social planner role can be summarized as maximizing $c(t)$ (as per say $y(t)$ and $\omega_i(t)$) by choosing the share of labour for the main production function and the skill-premium between the two sectors, or as per say $\delta(t)$, $x(t)$, and $w(t)$ with respect to their definition as detailed above.

$$\max_{\delta(t),x(t),w(t)} \int_t u(c(t))\, e^{-\rho t} dt \tag{2}$$

$$\begin{aligned} \dot{s}_x(t) &= -x(t) \\ s_x(t) &\geq 0 \\ s_x(0) &= \overline{s_{x0}} \textit{ (Initial condition)} \end{aligned}$$

$$\begin{aligned} \dot{s}_w(t) &= (1-\alpha)x(t) + (1-\beta)z(t) - w(t) \\ s_w(t) &\geq 0 \\ s_w(0) &= 0 \textit{ (Initial condition)} \\ l_y(t) &= \delta(t)(\mathcal{D}(s_w(t))) \\ l_z(t) &= (1-\delta(t))(\mathcal{D}(s_w(t))) \end{aligned}$$

- The current-value Hamiltonian with co-state variables $\lambda_{sx}(t)$ and $\lambda_{sw}(t)$ is:

$$\begin{aligned} \mathcal{H}(t) &= u(c(t)) - \lambda_{sx}(t)(x(t)) \\ &+ \lambda_{sw}(t)((1-\alpha)x(t) + (1-\beta)z(t) - w(t)) \end{aligned}$$

- Lagrangian with multipliers $\mu_\delta(t)$, $\mu_x(t)$, $\mu_{sx}(t)$, $\mu_w(t)$, and $\mu_{sw}(t)$

$$\max_{\delta(t),x(t),w(t)} \mathcal{L} = \mathcal{H} + \mu_\delta \delta + \mu_x x + \mu_{sx} s_x + \mu_w w + \mu_{sw} s_w$$

$$\begin{aligned}
\delta(t) &: u'(c(t))\Big(A_z(t)f'_{m(t)}h'_{\delta(t)} - f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)} + \mu_\delta(t) = 0 && (3)\\
x(t) &: u'(c(t))A_x(t)f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) && (4)\\
w(t) &: u'(c(t))A_z(t)f'_{m(t)}h'_{w(t)} + \lambda_{sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) + \mu_w(t) = 0 && (5)\\
s_x(t) &: \lambda_{sx}(t)\rho - \dot{\lambda_{sx}}(t) = \mu_{sx} && (6)\\
s_w(t) &: u'(c(t))\mathcal{D}'(s_w(t))\Big(A_z(t)f'_{m(t)}h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)}\mathcal{D}'(s_w(t)) + \mu_{sw}(t)\\
&= \lambda_{sw}(t)\rho - \dot{\lambda_{sw}}(t) && (7)
\end{aligned}$$

Complementary slackness:

$$\begin{aligned}
\frac{\partial\mathcal{L}(t)}{\partial\delta(t)} &= \delta(t)\geq 0, \delta(t)\geq 0, \delta(t)\,\delta(t)\geq 0 && (8)\\
\frac{\partial\mathcal{L}(t)}{\partial\mu_x(t)} &= x(t)\geq 0, \mu_x(t)\geq 0, \mu_x(t)\,x(t)\geq 0 && (9)\\
\frac{\partial\mathcal{L}(t)}{\partial\mu_{sx}(t)} &= s_x(t)\geq 0, \mu_{sx}(t)\geq 0, \mu_{sx}(t)\,s_x(t)\geq 0 && (10)\\
\frac{\partial\mathcal{L}(t)}{\partial\mu_w(t)} &= w(t)\geq 0, \mu_w(t)\geq 0, \mu_w(t)\,w(t)\geq 0 && (11)\\
\frac{\partial\mathcal{L}(t)}{\partial\mu_{sw}(t)} &= s_w(t)\geq 0, \mu_{sw}(t)\geq 0, \mu_{sw}(t)\,s_w(t)\geq 0 && (12)
\end{aligned}$$

For the next sections we consider:

$$\begin{aligned}
y(t) = f(A_y(t), m(t), l_y(t)) &= A_y(t)\,m(t)^\theta\,l_y(t)^{1-\theta} && (13)\\
z(t) = h(E(t), w(t), l_z(t)) &= E(t)\,w(t)^\nu\,l_z(t)^\gamma && (14)\\
u(c(t)) &= \frac{c(t)^{1-\sigma}-1}{1-\sigma} && (15)\\
\mathcal{D}(s_w(t)) &= L\frac{2}{\iota}\left(\frac{\iota}{2} - e^{-\frac{1}{s_w(t)}}\right) && (16)
\end{aligned}$$

$$\begin{aligned}
\delta(t) &: c(t)^{1-\sigma}\left(\frac{A_z(t)}{m(t)}\theta\gamma\frac{z(t)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\gamma(t)\frac{z(t)}{\delta(t)} + \mu_\delta(t) = 0 && (17)\\
x(t) &: c(t)^{1-\sigma}\theta\frac{A_x(t)}{m(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) && (18)\\
w(t) &: c(t)^{1-\sigma}\theta\nu\frac{A_z(t)}{m(t)}\frac{z(t)}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) + \mu_w(t) = 0 && (19)\\
s_x(t) &: \lambda_{sx}(t)\rho - \dot{\lambda_{sx}}(t) = \mu_{sx} && (20)\\
s_{wx}(t) &: \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(A_{z(t)}\theta\gamma\frac{z(t)}{m(t)} + (1-\theta)\right) + \lambda_{sw}(t)(1-\beta)\gamma z(t)\right) + \mu_{sw}(t)\\
&= \lambda_{sw}(t)\rho - \dot{\lambda_{sw}}(t) && (21)
\end{aligned}$$

This model is solved theoretically to provide theoretical predictions on the dynamics of the economy. In a second time, data at the level of the city of Rome could be used to test the predictions empirically.

4 An endogenous growth model without any damage function (*Model 1*)

In this section, the policy maker does not internalize the pollution. The model is analyzed without accounting for the damage function. Indeed, in most of the European cities, waste is not internalised as a negative externality. Such a model provides insights on the actual economic balance happening when no policy favoring circular economy prevails. When waste leads to no damage, the only incentive to have a circular sector is the scarcity of the resources. The only reason to dedicate labour to the recycling sector is to overcome the reduced availability of x(t).

Lemma 1 There is no apparent incentive for not extracting all the exhaustible resources before to start recycling, which means that we have either $m(t) = A_x x(t)$ for $t \leq T_r$, and $m(t) = A_z z(t)$ for all $t \geq T_r$. The proof of lemma 1 is in Appendix A.

In this first case when no damage is accounted for, the economy has thus only stages represented in Figure 9 and in Figure 10. In the first stage, the production sector utilizes all the resources to grow its economy and only once the stock of resources is fully depleted, the circular economy sector is introduced as a substitute to the natural exhaustible resources (the proof of this model can be found in Appendix B). This tendency could also be avoided if a high skill-prima favoring the circular sector would be set. However, this hypothesis could be validated only with the support of policy.

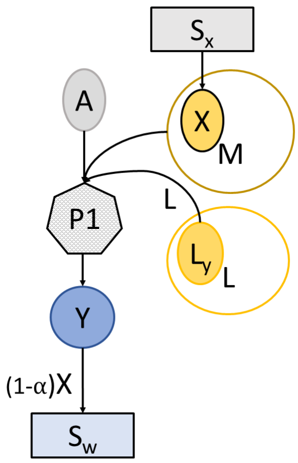

Figure 2: Economy at time $t_0 \leq t < T_r$

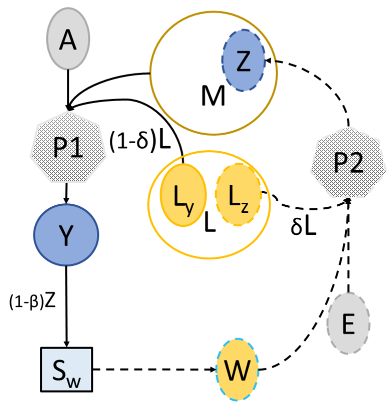

Figure 3: Economy at time $t \geq T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector. There is no damage function in this figure as it is not considered in this first step of the model.

The optimal trajectory of this model are given as follow (The details can be found in Appendix B.) We consider for any $t \geq 0$ that the growth of the exogenous productivity of sectors g_E and g_{A_y} being constant over time, and the productivity of the material A_x and A_z being constant.

General case:

$$T_r = \frac{1}{g_x} ln\Big(1 - \frac{s_x(T_r) - \overline{s}_{x0}}{\frac{x_0}{g_x}}\Big) \tag{22}$$

$$E(t) = \begin{cases} 0 & , t < T_r \\ E_{T_r} e^{g_E (t - T_r)} & , t \geq T_r \end{cases}$$

With $g_E = gx(1-\nu)$
understand why g_E is constrained

$$A_y(t) = A_{y0} e^{g_{A_y} t}$$

The consumption path is given by:

$$c(t) = \begin{cases} c_0 e^{g_c t} = c_0 e^{\frac{g_{A_y} - \theta\rho}{1-\theta(1-\sigma)} t} & , t < T_r \\ c_{T_r} e^{g_c^*(t-T_r)} = c_{T_r} e^{(g_{A_y} + \theta \frac{g_E}{1-\nu})(t-T_r)} & , t \geq T_r \end{cases}$$

where $c_0 = c(0) = A_y \left(A_x x(0)\right)^{\theta} L^{1-\theta}$
and $c_{T_r} = c(T_r) = A_y \left(A_z z(T_r)\right)^{\theta} ((1-\delta(T_r))L)^{1-\theta} = A_y \left(A_z \frac{w_{T_r}}{(2-\beta)\nu}\right)^{\theta} (\frac{1-\theta}{1-\theta+2\gamma\theta} L)^{1-\theta}$.
The share of labour directed to the recycling sector is given by:

$$\delta(t) = \begin{cases} 0 & , t < T_r \\ \frac{2\gamma\theta}{1-\theta+2\gamma\theta} & , t \geq T_r \end{cases}$$

The exhaustible resource use is given by:

$$x(t) = \begin{cases} x_0 e^{g_x t} = x_0 e^{\frac{(1-\sigma) g_{A_y} - \rho}{1-\theta(1-\sigma)} t} & , t < T_r \\ 0 & , t \geq T_r \end{cases}$$

The stock of exhaustible resource is given by:

$$s_x(t) = \begin{cases} \frac{x_0}{g_x}(1 - e^{g_x t}) + \overline{s}_{x0} & , t \leq T_r \\ s_x(T_r) & , t \geq T_r \end{cases}$$

The waste flow is given by:

$$w(t) = \begin{cases} 0 & , t < T_r \\ w(T_r) e^{g_w (t-T_r)} = w(T_r) e^{\frac{g_E}{1-\nu}(t-T_r)} & , t \geq T_r \end{cases}$$

The flow of recycled material is given by:

$$z(t) = \begin{cases} 0 & , t < T_r \\ \frac{w_{T_r}}{(2-\beta)\nu} e^{g_w (t-T_r)} = \frac{w_{T_r}}{(2-\beta)\nu} e^{\frac{g_E}{1-\nu}(t-T_r)} & , t \geq T_r \end{cases}$$

The material flow is given by:

$$m(t) = \begin{cases} A_x x_0 e^{g_x t} & , t < T_r \\ A_z z_{T_r} e^{g_w (t-T_r)} = \frac{A_z}{(2-\beta)\nu} w_{T_r} e^{\frac{g_E}{1-\nu}(t-T_r)} & , t \geq T_r \end{cases}$$

The stock of waste is given by:

$$s_w(t) = \begin{cases} (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = (1-\alpha)(\overline{s}_0 - s_x(T_r)) & , t < T_r \\ \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} e^{-g_w T_r} \left(e^{g_w (t-T_r)} - 1\right) + (1-\alpha)(\overline{s_{x0}} - s_x(T_r)) & , t \geq T_r \end{cases}$$

The co-state variables reads:

$$\lambda_{sw}(t) = \begin{cases} \lambda_{sw}(0)e^{\rho t} = \mu_w(t) & , t < T_r \\ \lambda_{sw}(T_r)e^{\rho(t-T_r)} & , t \geq T_r \end{cases}$$

with $\lambda_{sw}(T_r) = \frac{{c_{T_r}}^{1-\sigma}\theta(2-\beta)\nu}{w(T_r)}$ and $\lambda_{sw}(0) = \frac{{c_{T_r}}^{1-\sigma}\theta(2-\beta)\nu}{w(T_r)}e^{-\rho T_r}$

$$\lambda_{sx}(t) = \begin{cases} \theta\frac{{c_0}^{1-\sigma}}{x_0}e^{g_x T_r} + \lambda_{sw}(0)e^{\rho t}(1-\alpha) = \lambda_{sx}(0)e^{\rho t} & , t < T_r \\ 0 = {c_{T_r}}^{1-\sigma}e^{\left(g_{A_{y(t)}} + \theta\frac{g_{E(t)}}{1-\nu}\right)(1-\sigma)(t-T_r)}\theta\frac{A_x}{A_z}\frac{(2-\beta)\nu}{w(T_r)} + \lambda_{sw}(T_r)e^{\rho(t-T_r)}(1-\alpha) + \mu_x(t) & , t \geq T_r \end{cases}$$

$$\lambda_{sw}(T_r) = \frac{{c_{T_r}}^{1-\sigma}\theta\frac{A_x}{A_z}\frac{(2-\beta)\nu}{w(T_r)} + \mu_x(t)}{\alpha - 1}$$

with $\mu_x = \theta\left(\frac{{c_0}^{1-\sigma}}{x_0}e^{g_x T_r} - {c_{T_r}}^{1-\sigma}\frac{A_x}{A_z}\frac{(2-\beta)\nu}{w(T_r)}\right)$ and $\nu = \frac{1-\beta}{2-\beta}$

In this prediction with no damage, unfortunately, due to losses in the system, as soon as all natural resources are depleted, the economy will need to degrow. As this prediction is not advisable, it is of high importance to consider the damage induced by the stock of waste and to have policy favoring the circular sector to attract labour force, as it is demonstrated in the following model.

5 An endogenous growth model including a waste damage function (*Model 2 - not finished yet*)

When a damage function is considered, different mechanisms are expected to evolve and appear at different periods of time. There are supposedly four different periods that could be observed, depending on the labour allocation, see Figure 11 to Figure 14. Crucial determinants of these periods are the skill-premium between the two sectors, and the marginal cost of labour of the stock of waste $sw(t)$, or as per say, $D(sw)$ compared with circular sector. More details regarding these periods are provided in the Appendix D.

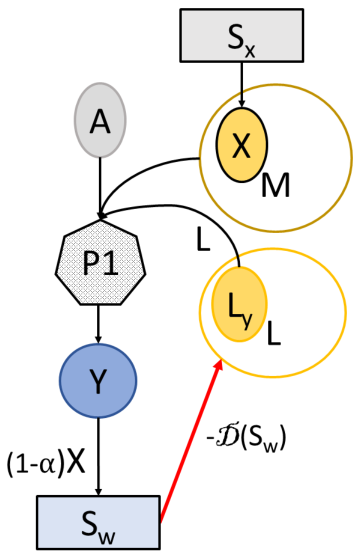

Figure 4: Economy at time $t_0 + \epsilon \leq t < T_{r+x}$

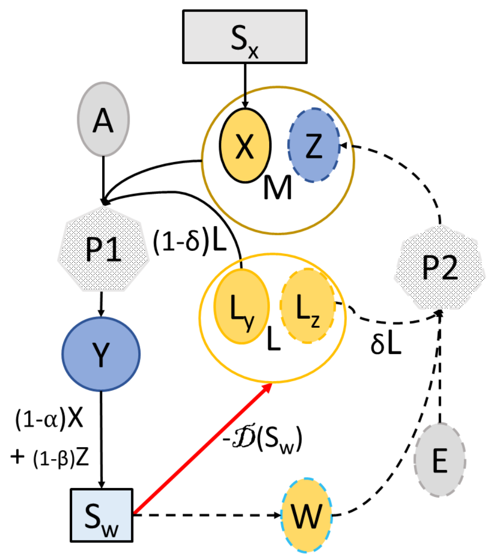

Figure 5: Economy at time $T_{r+x} \leq t < T_r$

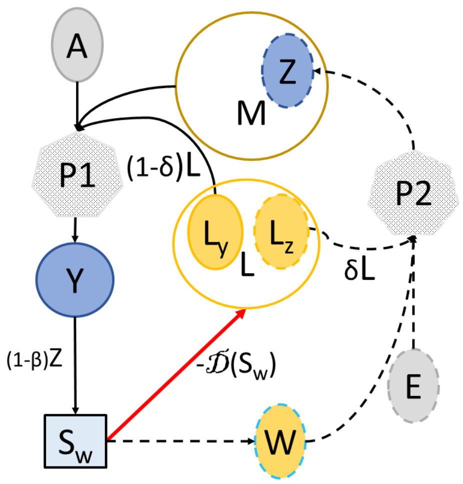

Figure 6: Economy at time $T_r \leq t < T_z$

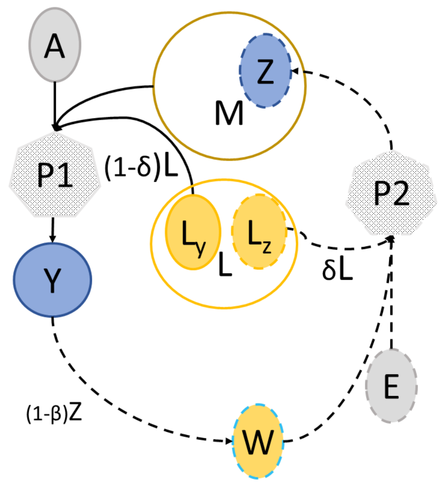

Figure 7: Economy at time $t \geq T_z$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the endogenous output variables are blue. The damage function is represented by a red arrow. The dashed lines represent the components of the recycling sector.

The first period, illustrated by Figure 11, is the traditional production model with damage function. As soon as the waste damage reduces the labour productivity under a certain threshold, it is more interesting for the labour and for the economy to diverge a part of the labour supply to the circular sector, leading to the model presented in Figure 5. The exact threshold for the circular economy to emerge, is to be determined by the model.

By considering the damage function, the market automatically balances and allows the circular sector to materialize before the exhaustion of the resources without any policy intervention. Such an event allows the economy to grow for a longer period of time as the exhaustible material is partially substituted by the circular sector. During the second period, for any $T_{r+x} \leq t < T_r$ the skill-premium is expected to be constant.

However, there might be a certain time T_r where the resources are fully depleted, leading to the model presented in Figure 13. This Third period might also be reached before the full depletion of exhaustible resources. Indeed, the more the circular sector uses materials, the fewer the waste stock impacts labour. Depending on the stock of waste and the skill-premium, consumer might prefer to supply labour to the circular sector over the production sector to improve their productivity. The economy might grow or degrow during this period. Depending on the skill-premium, consumers might also prefer to supply more labour to the production function to balance with the reduced amount of material available in the economy. For this third period, results are yet unpredictable and will depend on the specifications of the model.

Finally, as illustrated by Figure 14, once the stock of waste is fully depleted, the economy has no other choice than to degrow, as there is, for each period, fewer materials available in the economy.

An additional scenario presented by Figure 8 could consider to have simultaneously recycled materials and exhaustible resources, but the latter being in a proportion to only compensate for the losses of the system. In that very specific case, the materials input would be equals to $x(t) = \frac{\beta}{(1-\alpha)} z(t)$ and $z(t) = z(t+1)$. Such a model would consider no accumulation of waste and thus no damage on labour. Additionally, this model would reach a steady state of the economy (at least for a long period of time until full depletion of the exhaustible resources), postponing the degrowth. The optimum labour productivity and skill-premium will have to be estimated by the model.

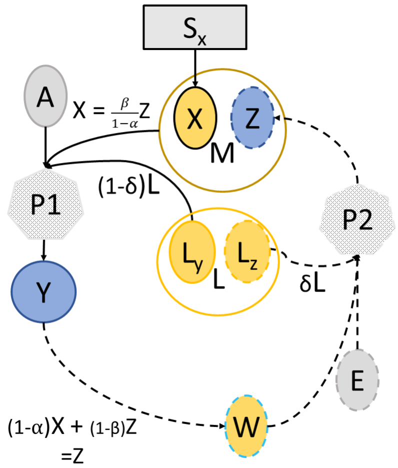

Figure 8: Fully circular model reaching the steady state

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector.

6 PARTS TO BE FURTHER DEVELOPED

6.1 Using Rome as a case study (*Data*)

The model predictions will then be empirically tested using the Italian city of Rome as a study case. Rome proposed an interesting study case which satisfies most of the assumptions set in the model. While being a city in a developed country, Rome is facing severe waste management issues[4]. Simultaneously, Rome is hastily developing its circular economy sector (Ghisellini & Ulgiati, 2020; *Italia che cambia*, 2018; *Atlas of the Circular Economy*, 2020), setting a perfect ground to test empirically the predictions of the theoretical model.

Using a capital city of the European Union to demonstrate the validity of the model predictions will further encourage policy makers worldwide to promote circular economy policies. Through such policies, local economic development will be promoted in a sustainable way (both for workers and the environment). My novel approach will complement the environmental economics literature on circular economy, expanding its scope to the field of labour productivity.

To form the panel data, economic levels of the city of Rome will be used. Outputs from the production sector and from the circular sector will be needed. To instrument the exhaustible material input, data on importation of raw materials will be considered. Similarly, data on waste recycled locally will be used as input for the circular sector. The losses will be estimated as the difference between waste and recycled materials. For the damage function, data on the health risk linked to improper waste management will be used. The labour productivity loss will be estimated based on the damage function and reevaluated with the data capturing the event of 2018, when the landfill nearby Rome accidentally ignited. The data on labour productivity and wage will be aggregated for the two sectors for the city of Rome. If no data is available it will be estimated from the

[4] *Rome in ruins: the ancient city lain waste* (2019)

country data based on average productivity of the city of Rome compared with the rest of Italy and on the working population. Most of the data mentioned is publicly available by the city of Rome (*Open Data - Roma Capitale*, n.d.).

Econometric dynamic modeling tools will be used following the literature initiated by Olley & Pakes (1992). The results from the dynamic modeling will be compared with the traditional OLS, GMM and IV models. External instruments will most probably have to be deployed to overcome issues linked with endogeneity. The econometric dynamic model should provide the most relevant results, nevertheless the GMM and external IV should provide similar range of results. Hopefully, the econometric model will confirm the predictions made by the theoretical model.

6.2 Policies to internalize waste externalities (*Expected results*)

The theoretical model will assess the potential role of circular economy to reduce waste linked externalities on labour. The model should provide different thresholds in terms of optimal labour supply, optimal skill-premium, optimal amount of materials (both recycled and exhaustible) to supply to the economy. From these optimal conditions, the study will evaluate the labour performance and timeline of the different periods of the economy (namely, when there is only the production sector, when the circular sector is an asset for the economy, when there are both supply of exhaustible and recycled material or only one of both etc.).

Additionally to the theoretical model, the empiric study will verify the prediction and provide specific results for the capital city of Italy, Rome. The results from the city of Rome will allow to extend the conclusions to the European Union. The study will potentially shed light on important mechanisms further supporting the circular economy sector. Policy makers are expected to embrace the results of this research and to develop instruments favoring the circular economy sector.

This research will combine both a novel theoretical model providing predictions on the impact of circular economy for labour productivity and empirical support to these predictions. The study will shed light on a new strand of literature related to the circular economy and will allow further research on the importance of circular economy in the field of labour economics.

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A Appendix 1: Without damage - Only one period at a time

When there is no damage function considered, x and z are never used at the same time.

Recycling phase

At time $t = T_{r+x}$ the second sector Z appears to overcome the scarcity of the stock of exhaustible resources. Z will also reduce the growth of the waste stock s_w. This second sector Z is labour intensive and uses waste w extracted from the stock of waste s_w to produce recycled material z to input in the production sector of the economy Y. Figure **??** illustrates the economy from T_{r+x} to T_r. Material used as inputs in the main production function are both exhaustible x extracted from our stock of exhaustible s_x and recycled z converted from waste w extracted from the stock of waste s_w.

$$\begin{aligned}
\delta(t) \neq 0 \quad &: \quad u'(c(t))\Big(A_z(t) f'_{m(t)} h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)} = 0 \\
&\iff c(t)^{1-\sigma}\Big(A_z(t)\frac{\theta}{m(t)}\frac{\gamma z(t)}{\delta(t)} - \frac{(1-\theta)}{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)\frac{\gamma z(t)}{\delta(t)} = 0 \quad (23) \\
x(t) \neq 0 \quad &: \quad u'(c(t))A_x(t) f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) = \lambda_{sx}(t) \\
&\iff c(t)^{1-\sigma}\left(A_x(t)\frac{\theta}{m(t)}\right) + \lambda_{sw}(t)(1-\alpha) = \lambda_{sx}(t) \quad (24) \\
w(t) \neq 0 \quad &: \quad u'(c(t))A_z(t) f'_{m(t)} h'_{w(t)} + \lambda_{sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0 \\
&\iff c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)}\frac{\nu z(t)}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\frac{\nu z(t)}{w(t)} - 1\right) = 0 \quad (25) \\
s_x(t) \neq 0 \quad &: \quad \frac{\dot{\lambda}_{sx}(t)}{\lambda_{sx}(t)} = \rho \quad (26) \\
s_w(t) \neq 0 \quad &: \quad \frac{\dot{\lambda}_{sw}(t)}{\lambda_{sw}(t)} = \rho \quad (27)
\end{aligned}$$

From (48) we have:

$$\lambda_{sw}(t) = c(t)^{1-\sigma}\theta\frac{A_z(t)}{m(t)}\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right) \quad (28)$$

but from (46) we have:

$$\lambda_{sw}(t) = c(t)^{1-\sigma}\theta\frac{A_z(t)}{m(t)}\frac{1}{(1-\beta)}\Big(1 - \frac{\delta(t)}{1-\delta(t)}\frac{m(t)(1-\theta)}{\gamma\theta A_z(t)z(t)}\Big) \quad (29)$$

Thus that gives:

$$\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right) = \frac{1}{(1-\beta)}\Big(1 - \frac{\delta(t)}{1-\delta(t)}\frac{m(t)(1-\theta)}{\gamma\theta A_z(t)z(t)}\Big) \quad (30)$$

Additionally, from equations (46), (47) and (48) we obtain:

$$\begin{aligned}
\lambda_{sx}(t) &= c(t)^{1-\sigma}\frac{\theta}{m(t)}\left(A_x(t) + (1-\alpha)A_z(t)\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right) \quad (31) \\
\delta(t) &= \left(\frac{u'(c(t))(1-\theta)c(t)}{\gamma z(t)\left(u'(c(t))A_z(t)\theta\frac{c(t)}{m(t)} + \lambda_{sw}(t)(1-\beta)\right)} + 1\right)^{-1} \quad (32) \\
\iff \delta(t) &= \left(\frac{(1-\theta)}{\gamma\theta\frac{z(t)A_z(t)}{m(t)}\left(1 + \left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right)} + 1\right)^{-1} \quad (33)
\end{aligned}$$

For (47) and (48) we have

$$\begin{aligned}(1-\sigma)g_c &= \rho + g_m && (34)\\ (1-\sigma)g_c &= \rho + g_m + (g_z - g_w)\left(\frac{\lambda_{sw}(t)}{c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)}} - 1\right) && (35)\end{aligned}$$

Thus, we have either $g_z = g_w$ or we have $\frac{\lambda_{sw}(t)}{c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)}} = 1 = \frac{h'_w(t)}{1-(1-\beta)h'_w(t)}$ which both lead to the same conclusion, (51) and (55) now reads:

$$\begin{aligned}\lambda_{sw}(t) &= c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)} && (36)\\ \lambda_{sx}(t) &= c(t)^{1-\sigma}\frac{\theta}{m(t)}\left(A_x(t) + (1-\alpha)A_z(t)\right) && (37)\end{aligned}$$

and:

$$\begin{aligned}z(t) &= \frac{1}{(2-\beta)\nu}w(t) \Longleftrightarrow g_z = g_w && (38)\\ \dot{s_w}(t) &= (1-\alpha)x(t) + w(t)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right) && (39)\\ \frac{1}{(2-\beta)\nu}w(t) &= E(t)L^{\gamma}\delta(t)^{\gamma}w(t)^{\nu} && (40)\\ \delta(t)^{\gamma} &= \frac{w(t)^{1-\nu}}{(2-\beta)\nu E(t)L^{\gamma}} \Longleftrightarrow g_{\delta} = \frac{g_w(1-\nu) - g_E}{\gamma} && (41)\end{aligned}$$

From (53) and (57) we get:

$$\begin{aligned}\delta(t) &= \left(\frac{(1-\theta)}{\gamma\theta\beta\frac{z(t)A_z(t)}{m(t)}} + 1\right)^{-1} = \left(\frac{(1-\theta)}{\gamma\theta\frac{z(t)A_z(t)}{m(t)}\left(1 + \left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right)} + 1\right)^{-1} && (42)\\ \beta &= \left(1 + \left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right) && (43)\end{aligned}$$

As we have $1 = \frac{h'_w(t)}{1-(1-\beta)h'_w(t)}$ thus, $\beta = 1$ which is not possible so we can not have at the period $x(t)$ and $z(t)$.

B Appendix 2: Details of the model when no damage function is accounted for

When waste leads to no damage, the only incentive to have a recycling function is the scarcity of the resource. The only reason to dedicate labour to the recycling sector is to overcome the reduced availability of x(t). Skipping the time index for convenience, and taking into account the assumptions on no loss, the current-value Hamiltonian with co-state variables $\lambda_{sx}(t)$ and $\lambda_{sw}(t)$ is:

$$\begin{aligned}\mathcal{H} &= u\Big(f\Big(A_y, A_x x + A_z h\big(E, w, \delta L\big), (1-\delta)L\Big)\Big)\\ &- \lambda_{sx} x\\ &+ \lambda_{sw}\Big((1-\alpha)x + (1-\beta)h\big(E, w, \delta L)\big) - w\Big)\end{aligned}$$

- Lagrangian with multipliers $\mu_\delta(t)$, $\mu_x(t)$, $\mu_{sx}(t)$, and $\mu_w(t)$

$$\max_{\delta(t),x(t),w(t)} \mathcal{L} = \mathcal{H} + \mu_\delta\delta + \mu_x x + \mu_{sx}s_x + \mu_w w + \mu_{sw}s_w$$

- Optimal conditions

$$\begin{aligned}\max_{\delta(t),x(t),w(t)} \mathcal{L} &= u\Big(f\Big(A_y, A_x x + A_z h\big(E, w, \delta L\big), (1-\delta)L\Big)\Big)\\ &- \lambda_{sx} x\\ &+ \lambda_{sw}\Big((1-\alpha)x + (1-\beta)h\big(E, w, \delta L)\big) - w\Big)\\ &+ \mu_\delta\delta + \mu_x x + \mu_w w + \mu_{sx}s_x + \mu_{sw}s_w \end{aligned} \tag{44}$$

$$\delta(t) : u'(c(t))\Big(A_z(t)f'_{m(t)}h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)} + \mu_\delta(t) = 0 \tag{45}$$

$$x(t) : u'(c(t))A_x(t)f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \tag{46}$$

$$w(t) : u'(c(t))A_z(t)f'_{m(t)}h'_{w(t)} + \lambda_{sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) + \mu_w(t) = 0 \tag{47}$$

$$s_x(t) : \lambda_{sx}(t)\rho - \dot{\lambda_{sx}}(t) = \mu_{sx} \tag{48}$$

$$s_w(t) : \lambda_{sw}(t)\rho - \dot{\lambda_{sw}}(t) = \mu_{sw} \tag{49}$$

$$\frac{\partial\mathcal{L}(t)}{\partial\delta(t)} = \delta(t) \geq 0, \delta(t) \geq 0, \delta(t)\delta(t) \geq 0 \text{ (complementary slackness)} \tag{50}$$

$$\frac{\partial\mathcal{L}(t)}{\partial\mu_x(t)} = x(t) \geq 0, \mu_x(t) \geq 0, \mu_x(t)x(t) \geq 0 \text{ (complementary slackness)} \tag{51}$$

$$\frac{\partial\mathcal{L}(t)}{\partial\mu_{sx}(t)} = s_x(t) \geq 0, \mu_{sx}(t) \geq 0, \mu_{sx}(t)s_x(t) \geq 0 \text{ (complementary slackness)} \tag{52}$$

$$\frac{\partial\mathcal{L}(t)}{\partial\mu_w(t)} = w(t) \geq 0, \mu_w(t) \geq 0, \mu_w(t)w(t) \geq 0 \text{ (complementary slackness)} \tag{53}$$

$$\frac{\partial\mathcal{L}(t)}{\partial\mu_{sw}(t)} = s_w(t) \geq 0, \mu_{sw}(t) \geq 0, \mu_{sw}(t)s_w(t) \geq 0 \text{ (complementary slackness)} \tag{54}$$

Lemma 1 There is no apparent incentive for not extracting all the exhaustible resources before to start recycling, which means that we have either $m(t) = A_x x(t)$ for $t \leq T_r$, and $m(t) = A_z z(t)$ for all $t \geq T_r$. The proof of the lemma 1 is in Appendice A.

Pre-recycling phase

At time $t = t_0$, the stock of exhaustible resource s_x equals its maximum $\overline{s}_0$. This stock might be deplete to a lower limit of 0. We are not sure yet that the economy will exhaust all the exhaustible resources available in this stock. The stock of labour l equals 1. All the others stocks are null as no waste has been produced yet. Following $t = t_0$, three periods are expected to occur. At time $t = t_0$, exhaustible resources are extracted from our stock s_x and input in the main production function. At this point in time, illustrated by figure 9, the waste stock s_w already starts to grow while the stock of exhaustible resource s_x diminishes. There is not yet any recycling. Thus for all $t_0 \leq t < T_r$:

$$
\begin{aligned}
\delta(t) = 0 \quad &: \quad c(t)^{1-\sigma}(1-\theta) = \mu_\delta(t) && (55)\\
x(t) \neq 0 \quad &: \quad u'(c(t))A_x(t)f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) = \lambda_{sx}(t) \\
&\iff \frac{c(t)^{1-\sigma}\theta}{x(t)} = \lambda_{sx}(t) - \lambda_{sw}(t)(1-\alpha) && (56)\\
w(t) = 0 \quad &: \quad \mu_w(t) = \lambda_{sw}(t) && (57)\\
s_x(t) \neq 0 \quad &: \quad \lambda_{sx}(t)\rho - \dot{\lambda}_{sx}(t) = 0 && (58)\\
s_w(t) \neq 0 \quad &: \quad \lambda_{sw}(t)\rho - \dot{\lambda}_{sw}(t) = 0 && (59)
\end{aligned}
$$

We can rewrite the FOC for all $t_0 \leq t < T_r$ such as:

$$
\begin{aligned}
\delta(t) &= z(t) = w(t) = 0 \\
\mu_\delta(t) &= c(t)^{1-\sigma}(1-\theta) && (60)\\
\lambda_{sx}(t) &= c(t)^{1-\sigma}\frac{\theta}{x(t)} + \lambda_{sw}(1-\alpha) = \lambda_{sx}(0)e^{\rho t} && (61)\\
\lambda_{sw}(t) &= \lambda_{sw}(0)e^{\rho t} = \mu_w(t) && (62)\\
\dot{s_x}(t) &= -x(t) && (63)\\
\dot{s_w}(t) &= (1-\alpha)x(t) && (64)\\
c(t) &= A_{y(t)}\left(A_{x(t)}x(t)\right)^\theta (L)^{1-\theta} && (65)\\
(1-\sigma)g_{c(t)} &= g_{x(t)} + \rho \iff g_c(t) = \frac{g_{A_Y} - \theta\rho}{1-\theta(1-\sigma)} && (66)\\
g_{x(t)} &= \frac{(1-\sigma)g_{A_{y(t)}} - \rho}{1-\theta(1-\sigma)} && (67)
\end{aligned}
$$

With $g_{c(t)} = \frac{\dot{c(t)}}{c(t)}$ and because for $t < T_r$ we have $g_{m(t)} = g_{x(t)} = \frac{\dot{x(t)}}{x(t)}$
For all $t_0 \leq t < T_r$, the flow variables read:

$$
\begin{aligned}
\delta(t) &= z(t) = w(t) = 0 && (68)\\
c(t) &= c_0 e^{\frac{g_{A_y}-\theta\rho}{1-\theta(1-\sigma)}t} && (69)\\
x(t) &= x_0 e^{g_x t} && (70)\\
m(t) &= A_x x_0 e^{\frac{(1-\sigma)g_{A_y}-\rho}{1-\theta(1-\sigma)}t} && (71)
\end{aligned}
$$

where $c_0 = c(0) = A_y\left(A_x x(0)\right)^\theta L^{1-\theta}$

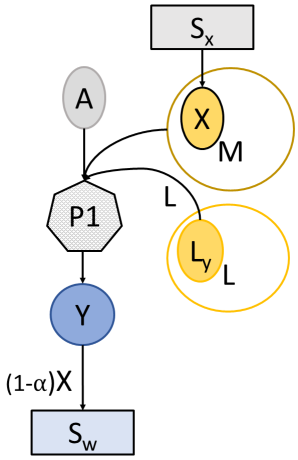


Figure 9: Economy at time $t_0 \leq t < T_r$

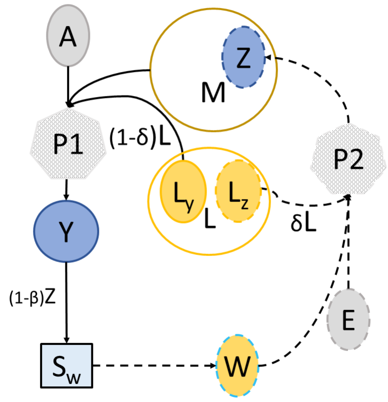


Figure 10: Economy at time $t \geq T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

The stock variables read:

$$s_x(t) = \frac{-x_0}{g_x} e^{g_x t} + S_{xT} \tag{72}$$

$$s_w(t) = (1-\alpha)(\frac{x_0}{g_x} e^{g_x t}) + S_{wT} \tag{73}$$

S_{xT} and S_{wT} being two constant to be defined.
We know from the initial condition that:

$$s_x(0) = \overline{s_{x0}} = \frac{-x_0}{g_x} + S_{xT} \iff S_{xT} = \overline{s_{x0}} + \frac{x_0}{g_x} \tag{74}$$

$$s_x(t) = \frac{x_0}{g_x}(1 - e^{g_x t}) + \overline{s_{x0}} \tag{75}$$

$$s_w(0) = 0 = (1-\alpha)(\frac{x_0}{g_x}) + S_{wT} \iff S_{wT} = -(1-\alpha)\frac{x_0}{g_x} \tag{76}$$

$$s_w(t) = (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) \tag{77}$$

And the co-state variables read:

$$\lambda_{sx}(t) = c(t)^{1-\sigma}\frac{\theta}{x(t)} + \mu_w(t)(1-\alpha) = \lambda_{sx}(0)e^{\rho t} \tag{78}$$

$$\lambda_{sw}(t) = \mu_w(t) = \lambda_{sw}(0)e^{\rho t} \tag{79}$$

Circular phase

After a time $t = T_r$, when no more exhaustible material is extracted, as depicts by figure 10. Exhaustible material are not used as input. There is two possible cases:

(1) either the stock of exhaustible resource is depleted and $s_x = 0$

$$\begin{aligned}
\delta(t) \neq 0 \quad &: \quad u'(c(t))\Big(A_z(t) f'_{m(t)} h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)} = 0 \\
&\iff c(t)^{1-\sigma}\left(\frac{\theta\gamma}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\frac{\gamma z(t)}{\delta(t)} = 0 \quad (80) \\
x(t) = 0 \quad &: \quad u'(c(t))A_x(t) f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \\
&\iff c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \quad (81) \\
w(t) \neq 0 \quad &: \quad u'(c(t))A_z(t) f'_{m(t)} h'_{w(t)} + \lambda_{sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0 \\
&\iff c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \quad (82) \\
s_x(t) = 0 \quad &: \quad \lambda_{sx}(t)\rho - \dot{\lambda}_{sx}(t) = \mu_{sx} \quad (83) \\
s_w(t) \neq 0 \quad &: \quad \frac{\dot{\lambda}_{sw}(t)}{\lambda_{sw}(t)} = \rho \quad (84)
\end{aligned}$$

(2) or the stock of resources is not depleted $s_x \neq 0$, but the economic rather prefer to input recycled material z instead of x for the main production sector. There is no argument in favor of exhaustible material which is fully substituted by the recycled material.

$$\begin{aligned}
\delta(t) \neq 0 \quad &: \quad u'(c(t))\Big(A_z(t) f'_{m(t)} h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{sw}(t)(1-\beta)h'_{\delta(t)} = 0 \\
&\iff c(t)^{1-\sigma}\left(\frac{\theta\gamma}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\frac{\gamma z(t)}{\delta(t)} = 0 \quad (85) \\
x(t) = 0 \quad &: \quad u'(c(t))A_x(t) f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \\
&\iff c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \quad (86) \\
w(t) \neq 0 \quad &: \quad u'(c(t))A_z(t) f'_{m(t)} h'_{w(t)} + \lambda_{sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0 \\
&\iff c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \quad (87) \\
s_w(t) \neq 0 \quad &: \quad \frac{\dot{\lambda}_{sw}(t)}{\lambda_{sw}(t)} = \rho \quad (88)
\end{aligned}$$

For $t > T_r$ we have

$$\begin{aligned}
x(t) &= 0 \\
z(t) &= \frac{1}{(2-\beta)\nu} w(t) \quad (89) \\
w(t) &= w_{Tr} e^{g_w(t-Tr)} \quad (90) \\
\lambda_{sw}(t) &= c(t)^{1-\sigma}\frac{\theta}{z(t)} = c(t)^{1-\sigma}\frac{\theta(2-\beta)\nu}{w(t)} = c(t)^{1-\sigma}\frac{\theta\nu}{w(t)\frac{(\beta-1)}{2-\beta} - 1} \quad (91) \\
\dot{s_w}(t) &= w(t)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right) \quad (92) \\
\delta(t) &= \left(\frac{2\gamma\theta}{1-\theta+2\gamma\theta}\right) \quad (93) \\
c(t) &= A_{y(t)}\left(\frac{A_{z(t)}w(t)}{(2-\beta)\nu}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2\gamma\theta}\right)^{1-\theta} \quad (94) \\
g_{z(t)} = g_{w(t)} = g_{m(t)} &= \frac{g_{E(t)}}{1-\nu} \quad (95) \\
g_{c(t)} &= g_{A_{y(t)}} + \theta\frac{g_{E(t)}}{1-\nu} \quad (96)
\end{aligned}$$

With

$$
\begin{aligned}
z(t) = \frac{w_{T_r}}{(2-\beta)\nu} e^{\frac{g_E}{1-\nu}(t-T_r)} &= E(t)\left(w(t)^{\nu}\right)(\delta L)^{\gamma} \\
\frac{w_{T_r}}{(2-\beta)\nu} e^{\frac{g_E}{1-\nu}(t-T_r)} &= E(t)\left(w_{T_r}^{\nu} e^{\nu \frac{g_E}{1-\nu}(t-T_r)}\right)(\delta L)^{\gamma} \\
w_{T_r} &= \sqrt[1-\nu]{(2-\beta)\nu E(T_r)(\delta L)^{\gamma}} \qquad (97) \\
w_{T_r} &= \sqrt[1-\nu]{(2-\beta)\nu E(t) e^{(\nu-1)\frac{g_E}{1-\nu}(t-T_r)}(\delta L)^{\gamma}} \qquad (98)
\end{aligned}
$$

From (16) and (77) we have:

$$
\begin{aligned}
E(T_r) &= E(t) e^{(\nu-1)\frac{g_E}{1-\nu}(t-T_r)} \\
E(t) &= E(T_r) e^{g_E(t-T_r)} \qquad (99)
\end{aligned}
$$

The flow variables read:

$$
\begin{aligned}
x(t) &= 0 \qquad (100) \\
c(t) &= c_{T_r} e^{(g_{A_y}+\theta\frac{g_E}{1-\nu})(t-T_r)} \qquad (101) \\
\delta(t) &= \frac{2\gamma\theta}{1-\theta+2\gamma\theta} \qquad (102) \\
z(t) &= \frac{w_{T_r}}{(2-\beta)\nu} e^{\frac{g_E}{1-\nu}(t-T_r)} \qquad (103) \\
m(t) &= \frac{A_z}{(2-\beta)\nu} w_{T_r} e^{\frac{g_E}{1-\nu}(t-T_r)} \qquad (104) \\
w(t) &= w(T_r) e^{\frac{g_E}{1-\nu}(t-T_r)} \qquad (105)
\end{aligned}
$$

where $c_{T_r} = c(T_r) = A_y\left(A_z z(T_r)\right)^{\theta}\left((1-\delta(T_r))L\right)^{1-\theta} = A_y\left(A_z \frac{w_{T_r}}{(2-\beta)\nu}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2\gamma\theta}L\right)^{1-\theta}$.
where $w(T_r) = \sqrt[1-\nu]{(2-\beta)\nu E(t)(\delta L)^{\gamma}}$
The stock variables read:

$$
\begin{aligned}
s_x(t) &= s_x(T_r) \qquad (106) \\
s_w(t) &= \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu}-1\right)}{g_w} e^{g_w(t-2Tr)} + S_{w2}(T) = \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu}-1\right)}{\frac{g_E}{1-\nu}} e^{\frac{g_E}{1-\nu}(t-2Tr)} + S_{w2}(T) \qquad (107)
\end{aligned}
$$

With $s_x(T_r)$ and $S_{w2}(T)$ endogeneous constant to be defined.
And the co-state variables read:

$$
\begin{aligned}
\lambda_{sx}(t) &= c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = c(t)^{1-\sigma}\frac{\theta}{z(t)}\left(\frac{A_x(t)}{A_z(t)} + (1-\alpha)\right) + \mu_x(t) \qquad (108) \\
\lambda_{sw}(t) &= c(t)^{1-\sigma}\frac{\theta}{z(t)} = \lambda_{sw}(T_r)e^{\rho(t-T_r)} = c(Tr)^{1-\sigma}\frac{\theta}{z(Tr)}e^{\rho(t-T_r)} \qquad (109) \\
\lambda_{sw}(Tr) &= c(Tr)^{1-\sigma}\frac{\theta}{z(Tr)} = \left(A_y\left(A_z\frac{w_{T_r}}{(2-\beta)\nu}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2\gamma\theta}L\right)^{1-\theta}\right)^{1-\sigma}\frac{\theta(2-\beta)\nu}{w_{T_r}} \qquad (110)
\end{aligned}
$$

To satisfy the transversality conditions, for all $t > Tr$:

$$
\lim_{t\to\infty}\lambda_{sx}(t)e^{-\rho(t-Tr)}s_x(t) = 0 \qquad (111)
$$

$$
\lim_{t\to\infty}\left(c(t)^{1-\sigma}\frac{\theta}{z(t)}\left(\frac{A_x(t)}{A_z(t)} + (1-\alpha)\right) + \mu_x(t)\right)e^{-\rho(t-Tr)}s_x(Tr) = 0
$$

For this condition to be valid, there are two possibilities: **either** $s_x(Tr) = 0$, thus the economy falls into the particular case where all the natural resource stock are consumed. Because stocks need to be continued we have:

$$s_x(T_r) = 0 = \frac{x_0}{g_x}(1 - e^{g_x T_r}) + \overline{s_{x0}} \quad (112)$$

$$T_r = \frac{1}{g_x} ln(1 + \frac{\overline{s_{x0}} g_x}{x_0}) \quad (113)$$

The only way this can be possible is for

$x_0 > -\overline{s_{x0}} g_x$

Or

$c(t)^{1-\sigma} \frac{\theta}{z(t)} \left(\frac{A_x(t)}{A_z(t)} + (1-\alpha) \right) = \mu_x(t)$ and thus for all $t > T_r$, $\lambda_{sx}(t) = 0$.

How to prove which one is the most probable?

At $t = T_r$ we have $s_w(t) = (1-\alpha)(\overline{s_{x0}} - s_x(T_r))$

$$(1-\alpha)(\overline{s_{x0}} - s_x(T_r)) = s_w(Tr) \quad (114)$$

$$(1-\alpha)(\overline{s_{x0}} - s_x(T_r)) = \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} e^{g_w(-Tr)} + S_{w2}(T)$$

$$S_{w2}(T) = (1-\alpha)(\overline{s_{x0}} - s_x(T_r)) - \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} e^{g_w(-Tr)} \quad (115)$$

$$\begin{aligned} s_w(t) &= \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} \left(e^{g_w(t-2T_r)} - e^{-g_w T_r} \right) + (1-\alpha)(\overline{s_{x0}} - s_x(T_r)) \quad (116) \\ &= \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} e^{-g_w T_r} \left(e^{g_w(t-T_r)} - 1 \right) + (1-\alpha)(\overline{s_{x0}} - s_x(T_r)) \end{aligned}$$

$$\begin{aligned} 0 &= \lim_{t\to\infty} \lambda_{sw}(t) e^{-\rho(t-Tr)} s_w(t) \quad (117) \\ &= \lim_{t\to\infty} c(Tr)^{1-\sigma} \frac{\theta}{z(Tr)} \left(\frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\nu} - 1\right)}{g_w} e^{-g_w T_r} \left(e^{g_w(t-T_r)} - 1 \right) + (1-\alpha)(\overline{s_{x0}} - s_x(T_r)) \right) \end{aligned}$$

I don't know what to get from this...

Optimal trajectories

The continuity of the current-value Hamiltonian gives:

$$\begin{aligned} \mathcal{H}(T_r^-) &= u\Big(A_y(A_x x(T_r^-))^\theta L^{1-\theta} \Big) \\ &- \lambda_{sx}(T_r^-) x(T_r^-) \\ &+ \lambda_{sw}(T_r^-) \Big((1-\alpha) x(T_r^-) \Big) \\ \mathcal{H}(T_r^+) &= u\Big(A_y(A_z E_{T_r^+} w(T_r^+)^\nu \delta^\gamma)^\theta (1-\delta)^{1-\theta} L^{1-\theta} \Big) \\ &+ \lambda_{sw}(T_r^+) \Big((1-\beta) E_{T_r^+} w(T_r^+)^\nu \delta^\gamma - w(T_r^+) \Big) \end{aligned}$$

If we consider L and A_y to be continued on T_r we have

$$\begin{aligned} x({T_r}^-)\Big(\lambda_{sx}({T_r}^-) - \lambda_{sw}({T_r}^-)(1-\alpha)\Big) &= u(c({T_r}^-)) \\ &- u(c({T_r}^+)) - \lambda_{sw}({T_r}^+)\Big((1-\beta)E_{{T_r}^+}w({T_r}^+)^\nu\delta^\gamma - w({T_r}^+)\Big) \end{aligned}$$

We know the exact value of $w({T_r}^+) = \sqrt[1-\nu]{(2-\beta)\nu E({T_r}^+)(\delta L)^\gamma}$ and $\lambda_{sw}({T_r}^+) = c({T_r}^+)^{1-\sigma}\frac{\theta(2-\beta)\nu}{w({T_r}^+)}$ with $c({T_r}^+) = A_{y({T_r}^+)}\left(\frac{A_{z({T_r}^+)}w({T_r}^+)}{(2-\beta)\nu}\right)^\theta\left(\frac{1-\theta}{1-\theta+2\gamma\theta}\right)^{1-\theta}$. So $\mathcal{H}({T_r}^+)$ is fully defined and we know its value.
As we also have $\Big(\lambda_{sx}({T_r}^-) - \lambda_{sw}({T_r}^-)(1-\alpha)\Big) = c({T_r}^-)^{1-\sigma}\frac{\theta}{x({T_r}^-)}$

$$\begin{aligned} x({T_r}^-)\Big(c({T_r}^-)^{1-\sigma}\frac{\theta}{x({T_r}^-)}\Big) &= u(c({T_r}^-)) - \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma}\theta &= u(c({T_r}^-)) - \mathcal{H}({T_r}^+) = \frac{c({T_r}^-)^{1-\sigma}-1}{1-\sigma} - \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma}\left(\frac{1-\theta(1-\sigma)}{1-\sigma}\right) &= \frac{1}{1-\sigma} + \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma} &= \frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big) \\ c({T_r}^-) &= \sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)} \\ A_y\left(A_x x({T_r}^-)\right)^\theta L^{1-\theta} &= \sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)} \\ x({T_r}^-)^\theta &= \frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_x^\theta L^{1-\theta}} \\ x({T_r}^-) &= \sqrt[\theta]{\frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_x^\theta L^{1-\theta}}} \\ \lambda_{sx}({T_r}^-) - \lambda_{sw}({T_r}^-)(1-\alpha) &= \frac{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)\theta}{\sqrt[\theta]{\frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_x^\theta L^{1-\theta}}}} \end{aligned}$$

There is no continuity condition on λ_{sx}, λ_{sx} is continued from $t = [0; T_r[$ and equals to 0 after T_r.
How to prove it?

C Appendix 3: When considering only the pre-recycling period with no damage

In the special case where we consider all the stock of exhaustible resource to be consumed, then we have, according the transversality conditions:

$$\lim_{t\to\infty} \lambda_{sx}(t)e^{-\rho t}s_x(t) = 0 \tag{118}$$

$$\lim_{t\to\infty} \lambda_{sx}(0)e^{\rho t}e^{-\rho t}\big(\frac{-x_0}{g_x}e^{g_x t} + S_{xT}\big) = 0$$

$$\lim_{t\to\infty} \lambda_{sx}(0)\big(\frac{-x_0}{g_x}e^{g_x t} + S_{xT}\big) = 0$$

For this condition to be valid, the only possibility is that $S_{xT} = 0$ ($g_x \leq 0$ and $\lambda_{sx}(0) \neq 0$), thus in this particular case where we use all the natural resource stock, we have:

$$s_x(t) = \frac{-x_0}{g_x}e^{g_x t} = \frac{-x_0}{g_x}e^{\frac{(1-\sigma)g_{A_y}-\rho}{1-\theta(1-\sigma)}t}$$

$$s_x(0) = \overline{s_{x0}} = \frac{-x_0}{g_x} \tag{119}$$

$$x(0) = x_0 = -\overline{s_{x0}}g_x \tag{120}$$

Similarly, we have to satisfy

$$\lim_{t\to\infty} \lambda_{sw}(t)e^{-\rho t}s_w(t) = 0 \tag{121}$$

We know from the initial condition that:

$$s_w(0) = 0 = (1-\alpha)(\frac{x_0}{g_x}) + S_{wT} \Longleftrightarrow S_{wT} = -(1-\alpha)\frac{x_0}{g_x} \tag{122}$$

$$s_w(t) = (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) \tag{123}$$

$$\lim_{t\to\infty} \lambda_{sw}(0)e^{-\rho t}e^{-\rho t}(1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = 0$$

$$\lim_{t\to\infty} \lambda_{sw}(0)(1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = 0 \tag{124}$$

For this condition to be valid, the only possibility is that $\lambda_{sw}(0) = 0$, thus in this case we have:

$$\lambda_{sw}(t) = \mu_w(t) = 0 \Longleftrightarrow \lambda_{sx}(t) = c(t)^{1-\sigma}\frac{\theta}{x(t)} = \lambda_{sx}(0)e^{\rho t}$$

$$\lambda_{sx}(0) = c(0)^{1-\sigma}\frac{\theta}{x(0)} = \Big(A_y\left(-A_x\overline{s_{x0}}g_x\right)^{\theta}L^{1-\theta}\Big)^{1-\sigma}\frac{\theta}{-\overline{s_{x0}}g_x} \tag{125}$$

$$\lambda_{sx}(t) = \Big(A_y\left(-A_x\overline{s_{x0}}g_x\right)^{\theta}L^{1-\theta}\Big)^{1-\sigma}\frac{\theta}{-\overline{s_{x0}}g_x}e^{\rho t} \tag{126}$$

D Appendix 4: Details of the model when a damage function is accounted for

Different mechanisms are expected to evolve and appear at different periods of time. There are supposedly four different periods that could be observed, depending on the labour allocation. A crucial determinant of these periods is the marginal cost of labour of the stock of waste s_w, or as per say, $\mathcal{D}(s_w)$ compared with the recycling sector. These periods are detailed in the following section. At time $t = t_0$, before any change has occurred, the stock of exhaustible resource s_x equals its maximum $\overline{s}_x$. This stock might be deplete to a lower limit of 0. We are not sure yet that the economy will exhaust all the exhaustible resources available in this stock. The stock of labour l equals 1. All the others stocks are null as no waste has been produced yet.

Pre-recycling phase

At time $t = t_0 + \epsilon$, exhaustible resources are extracted from our stock s_x and input in the main production function. At this point in time, illustrated by figure 11, the waste stock s_w already starts to grow and to negatively impact the productivity of labour. The labour available starts to reduce because of the negative externalities due to the waste stock. There is not yet any recycling. Thus for all $t_0 + \epsilon \leq t < T_{r+x}$:

$$\delta\,(t) = 0 \quad : \quad u'(c\,(t))(1-\theta)c(t) = \mu_\delta(t) \iff c(t)^{1-\sigma}(1-\theta) = \mu_\delta(t) \tag{127}$$

$$x\,(t) \neq 0 \quad : \quad u'(c\,(t))A_x(t)f'_{m(t)} + \lambda_{sw}(t)(1-\alpha) = \lambda_{sx}\,(t)$$

$$\iff \frac{c(t)^{1-\sigma}\theta}{x(t)} = \lambda_{sx}\,(t) - \lambda_{sw}(t)(1-\alpha) \tag{128}$$

$$w\,(t) = 0 \quad : \quad \mu_w(t) = \lambda_{sw}(t) \tag{129}$$

$$s_x\,(t) \neq 0 \quad : \quad \lambda_{sx}\,(t)\,\rho - \dot{\lambda_{sx}}\,(t) = 0 \tag{130}$$

$$s_w\,(t) \neq 0 \quad : \quad u'(c\,(t))\mathcal{D}'(s_w(t))f'_{l_y(t)} = \lambda_{sw}\,(t)\,\rho - \dot{\lambda_{sw}}\,(t)$$

$$\iff \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\Big(c(t)^{1-\sigma}(1-\theta)\Big) = \lambda_{sw}\,(t)\,\rho - \dot{\lambda_{sw}}\,(t) \tag{131}$$

For $t_0 + \epsilon \leq t < T_{r+x}$, s_w only leads to pollution. Thus λ_{sw} has to be negative. To match the slackness conditions, two μ_w need be such as: For $t_0 + \epsilon \leq t < T_{r+x}$, we have μ_{w1}:

$$w\,(t) = 0 \quad : \quad -\mu_{w1}(t) = \lambda_{sw}(t) \tag{132}$$

(133)

For $t \geq T_{r+x}$, we have μ_{w2}:

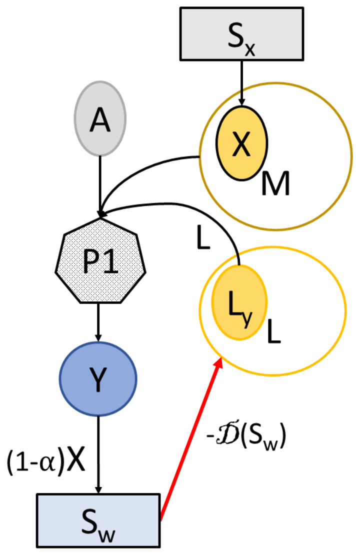
$$w\,(t) = 0 \quad : \quad \mu_{w2}(t) = \lambda_{sw}(t) \tag{134}$$

(135)

Recycling phase

with exhaustible injections

At time $t = T_{r+x}$ the second sector appears to reduce the growth of the waste stock s_w and to overcome both the negative externalities linked with the stock of exhaustible material s_x which is depleting and with the stock of waste s_w which reduces the available labour.



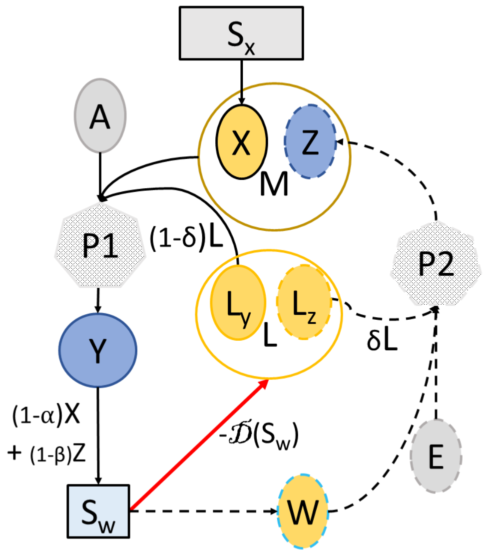


Figure 11: Economy at time $t_0+\epsilon \leq t < T_{r+x}$ Figure 12: Economy at time $T_{r+x} \leq t < T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

This second sector Z is labour intensive and uses waste w extracted from the stock of waste s_w to produce recycled material z to input in the main sector of the economy Y. Figure 12 illustrates the economy from T_{r+x} to T_r. Material used as inputs in the main production function are both exhaustible x extracted from our stock of exhaustible s_x and recycled z converted from waste w extracted from the stock of waste s_w. Differently from the model without damage, it is of higher interest to extract waste from the stock s_w as this stock has a negative impact on labour while the stock of exhaustible material s_x do not impact the labour. the period without recycling detailed above is supposedly shorter than the same period when no damage function is considered. During this period, the recycling sector possibly overcomes the waste-induced negatives externalities created by the exhausted material x on the labour l. These externalities are represented by the damage function $\mathcal{D}$, a function of the stock of waste from exhaustible resource s_w. The social planner has to decide how much labour l_z to dedicate to the recycling sector knowing that, without intervention on the stock of waste from exhaustible resource s_w, the amount of labour l available for the main production function decreases. At the same time, when labour l_z is dedicated to the recycling sector there is less labour l_y available to produce y from the main production. The social planner has to evaluate how much labour is polluted by a unit of exhaustible material compared with the labour needed to produce recycled material to substitute the exhaustible material from the waste stock s_w. The social planner has to evaluate the value of δ. The next section investigates further this bargain around labour allocation. For $T_{r+x} \leq t < T_r$, the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\frac{A_z(t)}{m(t)}\theta\gamma\frac{z(t)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\gamma(t)\frac{z(t)}{\delta(t)} = 0 \tag{136}$$

$$x(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_x(t)}{m(t)} + \lambda_{sw}(t)(1-\alpha) = \lambda_{sx}(t) \tag{137}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\nu\frac{A_z(t)}{m(t)}\frac{z(t)}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \tag{138}$$

$$s_x(t) \neq 0 \quad : \quad \lambda_{sx}(t)\rho - \dot{\lambda_{sx}}(t) = 0 \tag{139}$$

$$s_w(t) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(A_{z(t)}\theta\gamma\frac{z(t)}{m(t)} + (1-\theta)\right) + \lambda_{sw}(t)(1-\beta)\gamma z(t)\right)$$
$$= \lambda_{sw}(t)\rho - \dot{\lambda_{sw}}(t) \tag{140}$$

from (37)

$$\delta(t) = \left(\frac{u'(c(t))(1-\theta)c(t)}{\gamma z(t)\left(u'(c(t))A_z(t)\theta\frac{c(t)}{m(t)} + \lambda_{sw}(t)\right)} + 1\right)^{-1} \tag{141}$$

from (38), (39) and (40) we have:

$$\lambda_{sx}(t) = c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)}\left(\frac{A_x(t)}{A_z(t)} + \frac{\nu\frac{z(t)}{w(t)}}{1-\nu\frac{z(t)}{w(t)}}\right) \tag{142}$$

$$\lambda_{sw}(t) = c(t)^{1-\sigma}A_z(t)\frac{\theta}{m(t)}\left(\frac{\nu\frac{z(t)}{w(t)}}{1-\nu\frac{z(t)}{w(t)}}\right) \tag{143}$$

Circular phases

For the last period, illustrated by Figure 13, no more exhaustible material is extracted. Exhaustible material are not used as input.For these two periods, when $\beta < 1$, either $A_y(t)$ or $E(t)$ shall increase after $t = T_r$ to balance for the non-injection of exhaustible material and for the economy to stay constant or to grow. When $\beta = 1$, and both $A_y(t)$ and $E(t)$ stay constant after $t = T_r$, then the economy declines until $t = T_z$ when it reaches a steady state for all $t > T_z$.

without injection of exhaustible resources but with pollution from the stock of waste

There might be a time $t = T_r$, as depicted by figure 13, when the production function Y keep producing waste which accumulate in the stock of waste. This stock of waste damage the labour. For $T_r \leq t < T_z$ there are two possible cases: (1) either the stock of exhaustible resource is depleted and $s_x = 0$, the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\theta\gamma\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\gamma(t)\frac{z(t)}{\delta(t)} = 0 \tag{144}$$

$$x(t) = 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \tag{145}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \tag{146}$$

$$s_x(t) = 0 \quad : \quad \lambda_{sx}(t)\,\rho - \dot{\lambda}_{sx}(t) = \mu_{sx} \tag{147}$$

$$s_w(t) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(\theta\gamma + (1-\theta)\right) + \lambda_{sw}(t)(1-\beta)\gamma z(t)\right) = \lambda_{sw}(t)\,\rho - \dot{\lambda}_{sw}(t) \tag{148}$$

(2) There is no argument in favor of exhaustible material which is fully substituted by the recycled material. The economic rather benefits from the recycling sector than the exhaustible material and the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\theta\gamma\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\gamma(t)\frac{z(t)}{\delta(t)} = 0 \tag{149}$$

$$x(t) = 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \tag{150}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \tag{151}$$

$$s_x(t) \neq 0 \quad : \quad \lambda_{sx}(t)\,\rho - \dot{\lambda}_{sx}(t) = 0 \tag{152}$$

$$s_w(t) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(\theta\gamma + (1-\theta)\right) + \lambda_{sw}(t)(1-\beta)\gamma z(t)\right) = \lambda_{sw}(t)\,\rho - \dot{\lambda}_{sw}(t) \tag{153}$$

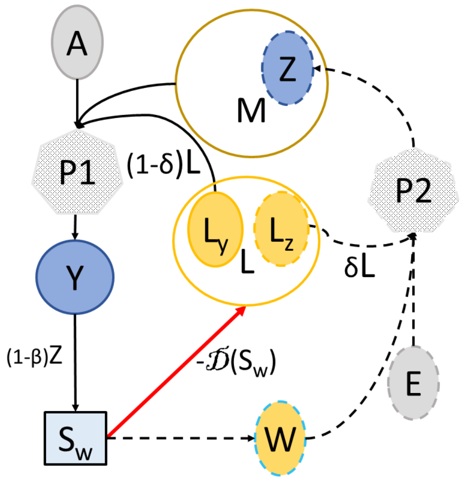


Figure 13: Economy at time $T_r \leq t < T_z$

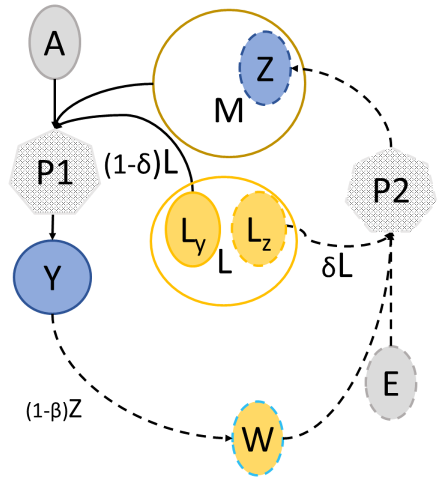


Figure 14: Economy at time $t \geq T_z$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

Special case: without injection of exhaustible resources and without pollution

There might be a time $t = T_z$, depicted by Figure 14 after when all goods are recyclable by design and the stock of waste s_w is depleted. This would mean that the stock of labour l does not endure any damage and $\mathcal{D}(s_w) = L$ as depicted by Figure 14. The goods which are not used by the consumers are directly reinjected in the second sector and recycled into z. For $T_r \leq t < T_z$, we have $\dot{s_w}(t) = 0 = (1-\beta)z(t) - w(t)$ or write differently $(1-\beta)z(t) = w(t)$ the first order conditions become:

$$\begin{aligned}
\delta(t) \neq 0 \quad &: \quad c(t)^{1-\sigma}\left(\theta\gamma\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)(1-\beta)\gamma(t)\frac{z(t)}{\delta(t)} = 0 \\
&\iff c(t)^{1-\sigma}\left(\theta\gamma\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{sw}(t)\gamma(t)\frac{w(t)}{\delta(t)} = 0 \qquad (154) \\
x(t) = 0 \quad &: \quad c(t)^{1-\sigma}\theta\frac{A_x(t)}{A_z(t)z(t)} + \lambda_{sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{sx}(t) \qquad (155) \\
w(t) \neq 0 \quad &: \quad c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left((1-\beta)\nu\frac{z(t)}{w(t)} - 1\right) = 0 \\
&\iff c(t)^{1-\sigma}\theta\nu\frac{1}{w(t)} + \lambda_{sw}(t)\left(\nu\frac{1}{w(t)} - 1\right) = 0 \qquad (156) \\
s_x(t) = 0 \quad &: \quad \lambda_{sx}(t)\rho - \dot{\lambda_{sx}}(t) = \mu_{sx} \qquad (157) \\
s_w(t) = 0 \quad &: \quad \mu_{sw} = \lambda_{sw}(t)\rho - \dot{\lambda_{sw}}(t) \\
& \qquad (158)
\end{aligned}$$

The next section evaluates the characteristics and the possible discontinuity between these periods. We investigate if at a certain point it becomes socially optimal to move from a linear economy (before $t = T_{r+x}$) to a fully circular economy (after $t = T_z$).

Long-run steady state

One of Figures **?? ?? ??**